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Bifurcations and Routes to Chaos in Wave-Structure Interaction Systems

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Wave-structure interaction systems are examined in this paper. These systems are characterized by a nonlinear restoring force and a coupled wave-structure drag and inertial exciting force. Stability analyses of system response define domains of primary and secondary resonances and reveal the existence of nonlinear solutions. Local and global bifurcations identify possible routes to chaotic motion and their controlling parameters. The analysis shows that period doubling and tangent bifurcations are enhanced by parametric excitation induced by the wave-structure coupling. Thus, complex dynamics recently uncovered numerically are obtained semianalytically allowing the identification of instabilities and their generating mechanisms.

Introduction

COMPLEX nonlinear and chaotic responses have been recently observed in various numerical models of compliant ocean systems.[1-4] These systems are characterized by an exciting force which couples wave and structure velocities and a nonlinear mooring restoring force. The restoring force includes material discontinuities and geometric nonlinearities associated with mooring line angles and has a unique equilibrium position. The exciting force includes a quadratic wave-structure drag component and a mixed displacement-velocity inertial component.

Although weakly nonlinear systems have been studied extensively from both classical[5] and modern approaches,[6] the scope of analyses of complex single equilibrium point (single potential well) systems has been limited. Examples of these systems are the hardening Duffing equation[7] and the motions of a wind-loaded structure.[8]

In past analysis of ocean systems, the restoring force of a mooring system was investigated with an equivalently linearized drag force.[1,2] Another example is the analysis of a quintic polynomial derived for the restoring moment of a rolling ship where the quadratic damping moment was linearized[9] or approximated by a mixed linear-cubic model.[10] These models assume local (temporal) wave kinematics which do not include the influence of convective acceleration. However, the influence of the nonlinear inertia force[11] cannot be neglected for large system response where the wave kinematics are evaluated at the displaced position of the structure.

The need for research of wave-structure interaction systems arises with the development of deep water compliant offshore structures which require a comprehensive understanding of strongly nonlinear systems designed for relatively large displacements. Nonlinear system behavior includes coexisting periodic (e.g., harmonic, subultraharmonic) and aperiodic (quasiperiodic, chaotic) solutions defined by different initial conditions. Consequently, system stability is governed by sensitivity to initial conditions and the mechanisms controlling the evolution of these states. Nonlinear systems that are sensitive to initial conditions exhibit transient or steady-state (strange attractor) chaotic motion that is unpredictable in a deterministic sense. Although existing ocean systems have not yet revealed chaotic behavior, there is a growing number of examples of physical systems exhibiting chaotic behavior.[12] Loss of predictability in nonlinear system behavior is of great importance and cannot be neglected in the analysis or design process. Therefore, the lack of systematic nonlinear analysis on one hand and the numerical evidence of complex nonlinear and chaotic response in idealized models on the other hand enhance the need for consistent analytical research of wave-structure interaction systems.

This paper describes a stability analysis performed on a nonlinear ocean system with a taut, symmetric mooring assembly. To investigate the complete nonlinear wave-structure coupling effect a system with an integrable (Hamiltonian) restoring force is chosen. The exciting force includes total (local plus convective) wave kinematics and the exact quadratic viscous drag component is retained.

System Model

The mooring system considered is modeled as a single degree-of-freedom, hydrodynamically damped and excited nonlinear oscillator (surge). The equation of motion is derived based on small-body motion under wave and weak current excitation[13] (see Fig. 1).

The restoring force R may contain a strong geometric nonlinearity depending on the magnitude of the mooring angle. The degree of nonlinearity can vary from a highly nonlinear two-point mooring system ($b = 0$) to an almost linear four-point mooring system ($b >> d$).

The exciting force consists of a drag component F_D and an inertial component F_I with frequency-independent coefficients. It is derived based on small-body theory which assumes that the presence of the structure does not affect the wave field, hence waves propagating past the structure remain unmodified. This approach can be justified for slender body motion in the vertical plane (surge, heave, pitch) where the wavelength is large compared to the beam of the structure.[14] The current is assumed to be weak, steady, inline with the waves and has a constant profile in the vertical direction.

The model is then described by the following equation of motion:

$$M\ddot{X} + C\dot{X} + R(X) = F_D(\dot{X},X,t) + F_I(\ddot{X},\dot{X},X,t) \quad (1a)$$

$$R = K[X + \mathrm{sgn}(X)b]\left\{1 - \sqrt{\frac{d^2 + b^2}{d^2 + [X + \mathrm{sgn}(X)b]^2}}\right\} \quad (1b)$$

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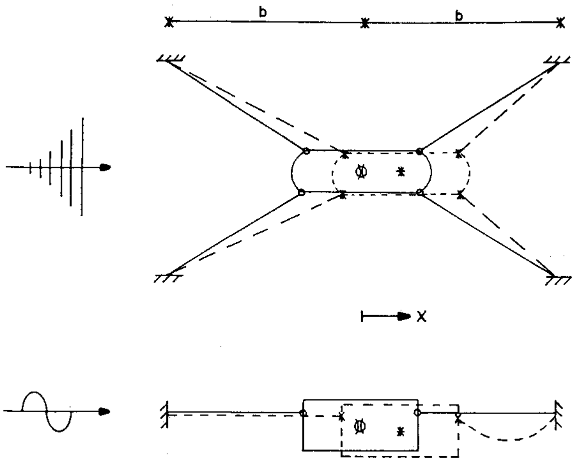


Fig. 1 Mooring assembly.

$$F_D = (1/2)\rho A_p C_d(U - \dot{X})|U - \dot{X}| \quad (1c)$$

$$F_I = \rho \forall(1 + C_a)\left[\frac{\partial U}{\partial t} + (U - \dot{X})\frac{\partial U}{\partial X}\right] - \rho \forall C_a \ddot{X} \quad (1d)$$

$$U = U_0 + \omega a U_1 \cos(kX - \omega t) \quad (1e)$$

and

- M, C, K — system mass, damping and stiffness [$K = 2EA_c/\sqrt{(d^2 + b^2)}$, EA_c elastic cable force]
- C_d, C_a — hydrodynamic viscous drag and added mass coefficients
- $A_p, \forall$ — projected drag area and displaced volume
- U_0, U_1 — current magnitude and depth (z) parameter [$U_1 = \cosh k(z + h)/\sinh(kh)$]
- a, ω, k — wave amplitude, frequency, and number ($\omega^2 = gk \tanh kh$)
- ρ, g, h — water mass density, gravitational acceleration, and water depth

Note that $(\cdot)$ is differentiation with respect to time and sgn (X) denotes the sign of X.

The expansion of the restoring force in a least-square sense yields an odd polynomial. Rearranging and normalizing ($x = kX$, $\theta = \omega t$) the equation of motion, Eq. (1a), results with the following first-order autonomous system:

$$\dot{x} = y, \qquad \dot{y} = -R(x) - \gamma y + F_D(x,y,\theta) + F_I(x,y,\theta)$$

$$\dot{\theta} = \omega \quad (2a)$$

where

$$R(x) = \sum_n \alpha_n x^n, \qquad n = 1, 3, 5, \ldots, N \quad (2b)$$

$$F_D(x, y, \theta) = (\mu\delta/\omega^2)(u - y)|u - y| \quad (2c)$$

$$F_I(x, y, \theta) = \mu[1 - (1/\omega)(u - y)]\dot{u} \quad (2d)$$

$$u(x, \theta) = f_0 + \omega f_1 \cos(x - \theta)$$

$$\dot{u}(x, \theta) = \omega^2 f_1 \sin(x - \theta) \quad (2e)$$

$$\alpha_n = \frac{K_n}{M + \rho\forall C_a}, \qquad K_n = K_n(K, kb, kd)$$

$$\gamma = \frac{C}{M + \rho\forall C_a}$$

$$\delta = \frac{1}{2}\frac{C_d}{1 + C_a}\frac{A_p}{\forall} g \tanh kh$$

$$\mu = \frac{\rho\forall(1 + C_a)}{M + \rho\forall C_a}$$

$$f_0 = kU_0, \qquad f_1 = kaU_1$$

Note that $\mu \geq 1$ (buoyancy), $f_1 < 1$ [$ka < \pi/7$ (limiting wave steepness), $U_1 \leq 1$ ($kh > \pi/10$ deep water)], $f_0 < \omega f_1$, $\delta < 1$ and $\gamma << 1$ (structural damping).

The degree of nonlinearity in the restoring force is characterized by the magnitude of the amplitudes α_n. The highly nonlinear two-point mooring system ($b = 0$) lacks a linear term ($\alpha_1 = 0$), whereas the weakly nonlinear four-point system is described with decreasing coefficients and is limited by a linearized system ($\alpha_1 = 1$, $\alpha_{n>1} = 0$).

As the normalized displacement ($x = kX$) is not large, the wave kinematics described in Eq. (2e) can be represented by the following finite trigonometric series expansion:

$$u(x, \theta) = f_0 + \omega f_1 \sum_n \left[\frac{x^{n-1}}{(n-1)!}\cos\theta + \frac{x^n}{n!}\sin\theta\right] \quad (3a)$$

$$\dot{u}(x, \theta) = \omega^2 f_1 \sum_n \left[\frac{x^n}{n!}\cos\theta - \frac{x^{n-1}}{(n-1)!}\sin\theta\right] \quad (3b)$$

where $n = 1, 3, 5, \ldots, N$.

Substitution of the trigonometric series, Eqs. (3), in the system Eq. (2a), enables isolation of nonlinear terms and identification of the controlling parameters governing the response in the following system representation:

$$\dot{x} = y, \qquad \dot{y} = F_1(x, \theta) + F_2(y, xy, \theta), \qquad \dot{\theta} = \omega \quad (4a)$$

where

$$F_1(x, \theta) = \sum_l [B_l + K_l(\theta)]x^l \quad (4b)$$

$$F_2(y, xy, \theta) = \sum_l \Gamma_l y^l + \left[\sum_l \Lambda_l(\theta)x^l\right]y \quad (4c)$$

and $l = 0, 1, 2, 3, \ldots, L$. [See Appendix for parameter detail ($L = 3$)]. Note that F_1 ($l = 0$) contributed a bias (B_0) and external excitation [$K_0(\theta)$], whereas both $F_1(l \geq 1)$ and $F_2(l \geq 0)$ include parametric excitation $\{h(\theta)x^1, h(\theta) : [K(\theta), \Lambda(\theta)], h(\theta) = \Sigma \cos(j\theta + \phi_j), j = 1, 2, 3, \ldots, L - 1\}$.

The governing system nonlinearities ($L = 3$) are quadratic (x^2, xy, y^2) and cubic (x^3). Furthermore, even the linearized mooring system ($\alpha_1 = 1$, $\alpha_{1>1} = 0$) that is subjected to small excitation ($\cos x \to 1$, $\sin x \to x$) retains the quadratic nonlinearities (y^2, xy) and the parametric excitation. Thus, wave-structure interaction systems are shown to be nonlinear, coupled, parametrically excited systems and are anticipated to exhibit complex dynamics[5,15,16] and chaotic motions.[8,17,18]

Stability Analysis

The global stability of the system is investigated by using a Lyapunov function approach.[19] A weak Lyapunov function [$V(x, y)$] can be found for the undamped and unforced Hamiltonian system [$V(x, y) = \frac{1}{2}y^2 + P(x)$ where $P(x) = \int R(x)\mathrm{d}x$]. Thus, the origin [$(x, y)_e = (0, 0)$] is neutrally stable [$V(0, 0) = 0$, $\mathrm{d}V/\mathrm{d}t \equiv 0$]. Modification of $V(x, y)$ to account for structural damping and choosing ν sufficiently small ($0 < \nu < \gamma$) results in the following Lyapunov function:

$$V(x, y) = (1/2)y^2 + P(x) + \nu[xy + (1/2)\gamma x^2] \quad (5a)$$

for which

$$\dot V(x, y) = -\nu[xR(x)] - (\gamma - \nu)y^2 \tag{5b}$$

The strong Lyapunov function, Eq. (5a), describes in the phase plane (x, y) an asymptotically stable [$V(x, y)$ positive definite, $dV/dt \le 0$], hyperbolic fixed point (sink) at the origin. Excitation by current alone ($f_0 \neq 0$, $f_1 = 0 \Rightarrow F_I = 0$) introduces a bias, but does not change the characteristics of the unforced system. However, with the addition of wave excitation, the sink becomes a hyperbolic closed orbit (limit cycle). The limit cycle loses the circularity of the sink but is anticipated by the invariant manifold theorem to retain its stable characteristics.[6] Although this result insures that solutions remain bounded for small excitation ($|F_D|, |F_I| << 1$), investigation of the influence of larger excitation can only be performed by local analysis.

Local stability is determined by considering the following perturbed solution:

$$x(t) = x_0(t) + \varepsilon(t) \tag{6a}$$

$$y(t) = y_0(t) + \eta(t) \tag{6b}$$

where $[x_0(t), y_0(t)]$ is an approximate solution and $[\varepsilon(t), \eta(t)]$ is a small variation.

The approximate solution can be obtained by a variety of methods.[5] Two examples are the method of generalized averaging for the weakly nonlinear system[20] and the method of harmonic balance for a system with a larger nonlinearity.[21] The latter method is chosen as it assumes a solution form which allows for the appearance of even harmonics in a low-order approximation.

$$x_0 \cong A_0 + \sum_i A_{i/m} \cos\left(i\frac{\theta}{m} + \Phi_{i/m}\right) \tag{7a}$$

$$y_0 \cong \frac{\omega}{m}\sum_i iA_{i/m} \sin\left(i\frac{\theta}{m} + \Phi_{i/m}\right) \tag{7b}$$

where A_0, $A_{i/m}$, $\Phi_{i/m}$ are solution amplitudes and phases, I is the order of approximation ($i = 1, 2, 3, \ldots, I$), and M is the order of subharmonic ($m = 1, 2, 3, \ldots, M$).

The unknown amplitudes and phases are obtained by transferring the original nonlinear system, Eq. (2a), into a finite nonlinear set of algebraic equations [$G_j(A_0, A_{i/m}, \Phi_{i/m})$; $j = 1, 2, \ldots, 2I + 1$]. Solution of the set with an iterative Newton-Raphson procedure results in a frequency-response relationship ($\omega - A_m$) (see Fig. 2). An unsymmetric solution

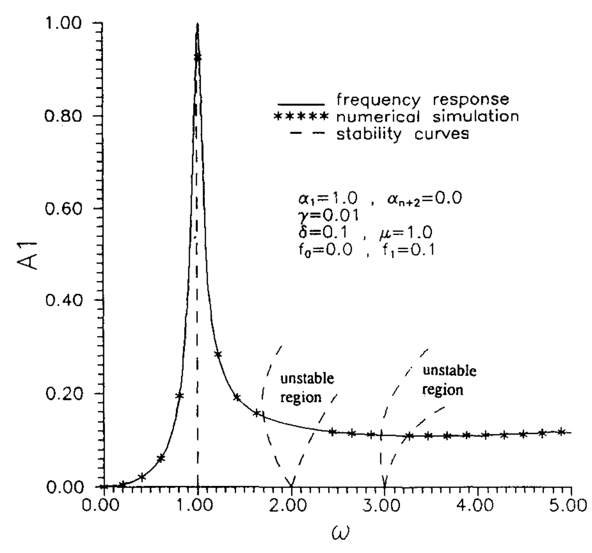


Fig. 2 Stability diagram.

includes even and odd harmonics [$x_0(t)$, $y_0(t) \neq x_0(t + mT/2)$, $y_0(t + mT/2)$; $T = 2\pi/\omega$], whereas a symmetric solution consists of only odd harmonics.

Substituting the solution, Eq. (6), in the equations of motion, Eq. (2a), and simplifying leads to the following nonlinear variational equations:

$$\dot\epsilon = \eta, \qquad \dot\eta = D(\epsilon, \eta; x_0, y_0) + G(\epsilon, \eta; x_0, y_0) \tag{8a}$$

where

$$D(\varepsilon, \eta) = -\gamma\eta + (\mu\delta/\omega^2)[(u - y)|u - y| - (u_0 - y_0)|u_0 - y_0|] \tag{8b}$$

$$G(\varepsilon, \eta) = -\sum_n \alpha_n(x^n - x_0^n) + \mu\{(\dot u - \dot u_0) - (1/\omega)[(u - y)\dot u - (u_0 - y_0)\dot u_0]\} \tag{8c}$$

and

$$u = f_0 + \omega f_1 \cos(x_0 + \varepsilon - \theta), \qquad u_0 = u(\varepsilon = 0)$$

$$\dot u = \left(1 - \frac{\eta}{\omega}\right)\omega^2 f_1 \sin(x_0 + \varepsilon - \theta), \qquad \dot u_0 = \dot u(\varepsilon, \eta = 0)$$

Linearizing Eqs. (8b) and (8c) yields the following second-order linear ordinary differential equation with periodic coefficient functions $H_{1,2}[x_0(t), y_0(t)] = H_{1,2}[x_0(t + T), y_0(t + T)]$:

$$\dot\varepsilon = \eta, \qquad \dot\eta = H_1(x_0, y_0)\eta + H_2(x_0, y_0)\varepsilon \tag{9a}$$

where

$$H_1 = -\gamma - 2(\mu\delta/\omega^2)|u_0 - y_0| + (\mu/\omega^2)(u_0 - y_0)\dot u_0 \tag{9b}$$

$$H_2 = -\sum_n n\alpha_n x_0^{n-1} - 2(\mu\delta/\omega^3)|u_0 - y_0|\dot u_0 + \mu\{(u_0 - f_0)[\omega - (u_0 - y_0)] + (1/\omega^2)\dot u_0^2\} \tag{9c}$$

Substituting the approximate solution, Eq. (7), in Eqs (9b) and (9c) and expanding $H_{1,2}(x_0, y_0)$ in a Fourier series $H_{1,2}(\theta)$ leads to the following generalized Hill's variational equation:

$$\dot\varepsilon = \eta, \qquad \dot\eta = H_1(\theta)\eta + H_2(\theta)\varepsilon \tag{10a}$$

where $H_{1,2}$ are calculated from the following:

$$H_1 = \zeta_0 + \sum_j \zeta_{Cj/m} \cos\left(j\frac{\theta}{m}\right) + \zeta_{Sj/m} \sin\left(j\frac{\theta}{m}\right) \tag{10b}$$

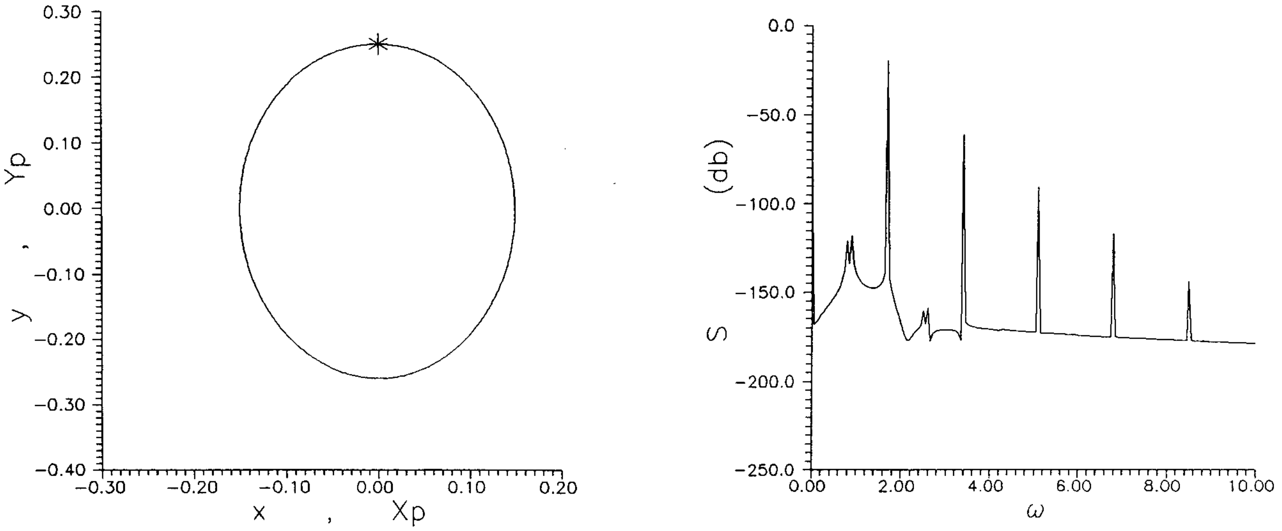
$$H_2 = \lambda_0 + \sum_j \lambda_{Cj/m} \cos\left(j\frac{\theta}{m}\right) + \lambda_{Sj/m} \sin\left(j\frac{\theta}{m}\right) \tag{10c}$$

and $a_j = (\zeta_{Cj}, \lambda_{Cj})$, $b_j = (\zeta_{Sj}, \lambda_{Sj})$ are Fourier coefficients calculated from H_1 and H_2, respectively:

$$a_{j/m} = \frac{1}{\pi}\int_{-\pi}^{\pi} H\left(\frac{\theta}{m}\right)\cos\left(j\frac{\theta}{m}\right) d\theta$$

$$b_{j/m} = \frac{1}{\pi}\int_{-\pi}^{\pi} H\left(\frac{\theta}{m}\right)\sin\left(j\frac{\theta}{m}\right) d\theta$$

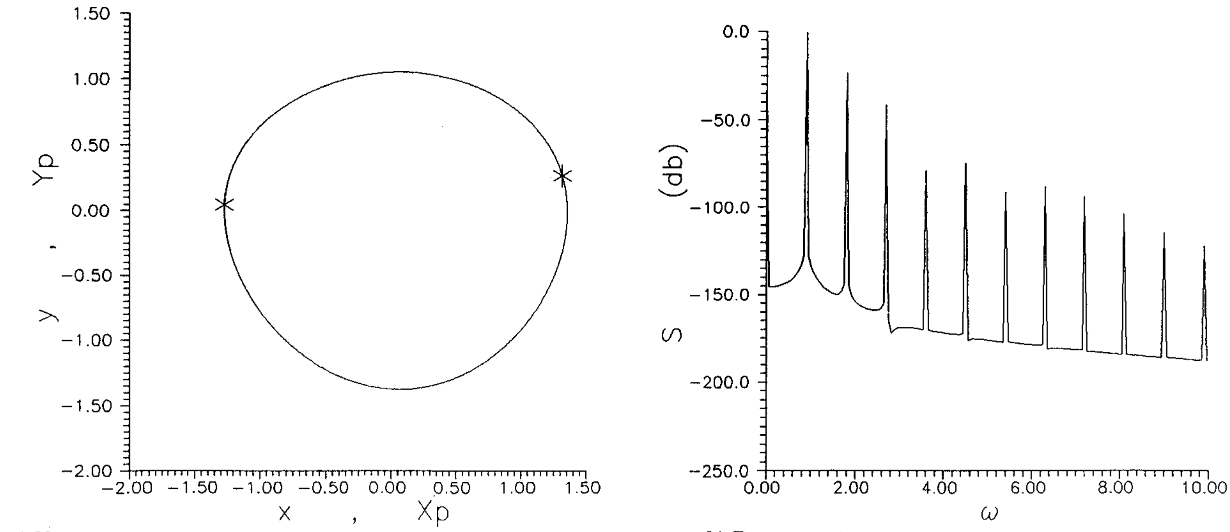
The particular solution to Eq. (10a) is $\varepsilon = \exp(\xi t)Z(t)$. Application of Floquet theory[22] yields two solution forms [$Z(t) = Z(t + mT)$, $Z(t) = Z(t + 2mT)$] which are due to the odd and even terms, respectively, in Eq. (7). Thus, two unstable regions are defined. The first unstable region [$Z(t) = Z(t + mT)$] is identified by the even terms ($m = 1 : j = 2$,



a) Nondimensional phase plane and Poincaré map

b) Power spectra

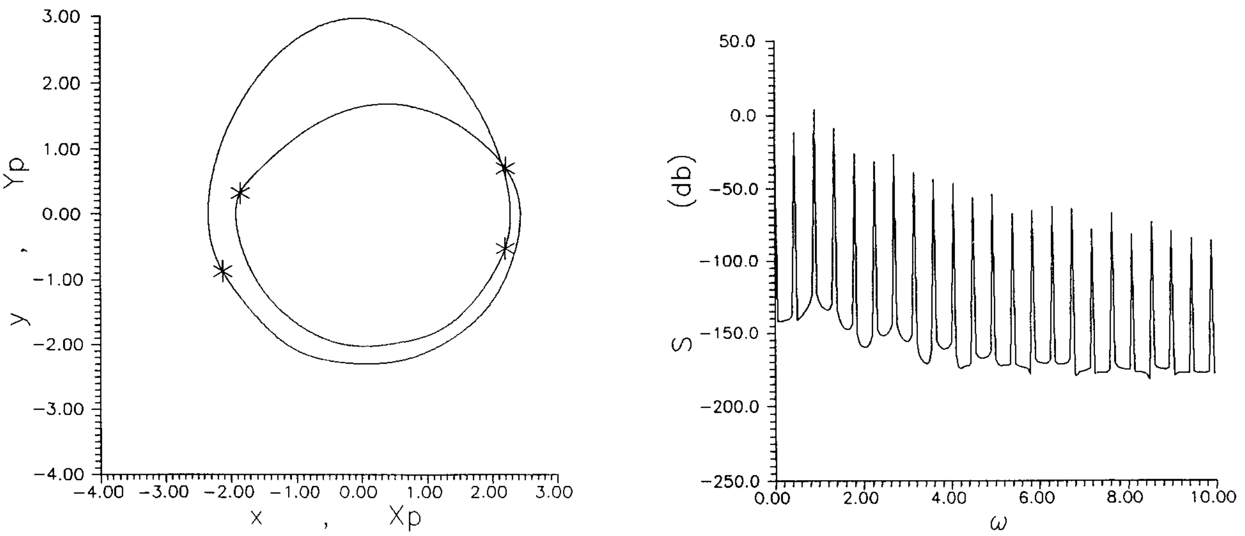
Fig. 3 Harmonic ($m = 1$) solution [$\omega = 1.6$].



a) Nondimensional phase plane and Poincaré map

b) Power spectra

Fig. 4 Subharmonic ($m = 2$) solution [$\omega = 1.9$].



a) Nondimensional phase plane and Poincaré map

b) Power spectra

Fig. 5 Subharmonic ($m = 4$) solution [$\omega = 1.7$].

4, 6, . . .) in Eq. (10a) and coincides with the vertical tangent points of a primary resonance on the frequency-response curve. The second unstable region $[Z(t) = Z(t \pm 2mT)$ is identified by the odd terms $(m = 1 : j = 1, 3, 5, \ldots)$ and reveals a secondary resonance which consists of a period-doubled solution.

The boundaries of the unstable regions can be obtained by applying the method of harmonic balance to the Hill's equation, Eq. (10a), for $\varepsilon(t)$ at the stability limit ($\xi = 0$). This is obtained by substituting the following solution form, Eq. (11), into Eq. (10a) and matching harmonic terms.

$$\varepsilon\left(\frac{\theta}{m}\right) = b_0 + \sum_j b_{j/m} \cos\left(j\frac{\theta}{m} + \phi_{j/m}\right)$$

$$Z(t + mT) \tag{11a}$$

$$\varepsilon\left(\frac{\theta}{2m}\right) = \sum_j b_{j/2m} \cos\left(j\frac{\theta}{2m} + \phi_{j/2m}\right)$$

$$Z(t + 2mT) \tag{11b}$$

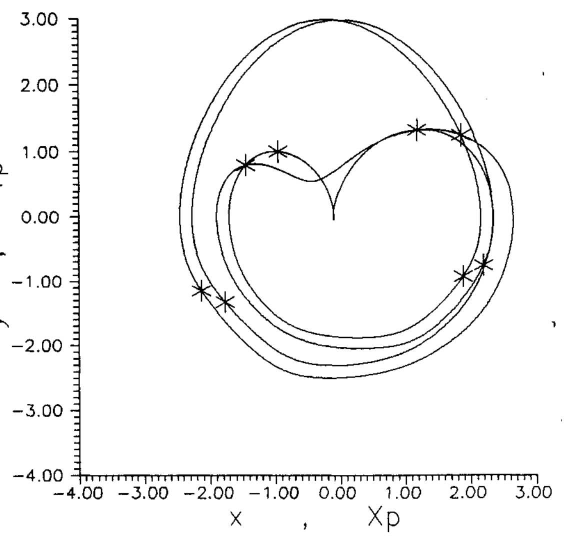
The condition for a nonzero solution results in a determinant $[\Delta(\omega^2) = 0]$ from which two hyperbolic stability curves defining the unstable region $[\Delta(\omega^2) < 0$ for $\nu > 0]$ are obtained (see Fig. 2).

$$\Delta(\omega^2) = \left[\lambda_0 - \left(\frac{\omega}{m}\right)^2\right]^2 + \left(\zeta_0\frac{\omega}{m}\right)^2 - \frac{1}{4}\left[\left(\lambda_{2C} - \frac{\omega}{m}\zeta_{2S}\right)^2 + \left(\lambda_{2S} + \frac{\omega}{m}\zeta_{2C}\right)^2\right] = 0 \tag{12}$$

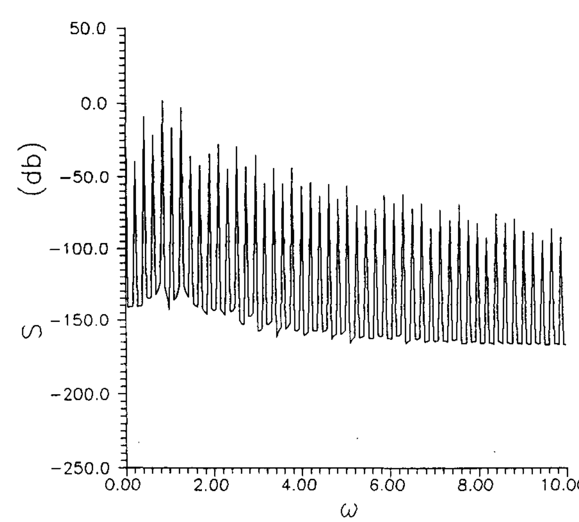
Intersection of approximate ($j = 1$) stability curves with the frequency-response curve defines in parameter space the domain of stability loss of the T periodic solution.

Bifurcations and Routes to Chaos

The variational equation, Eq. (10a), reveals regions where the mT periodic solution loses its stability. The transition to and from these steady states is defined by saddle-node (tangent) and period doubling (flip) bifurcations. These bifurcations are local and occur at the boundaries of the stability

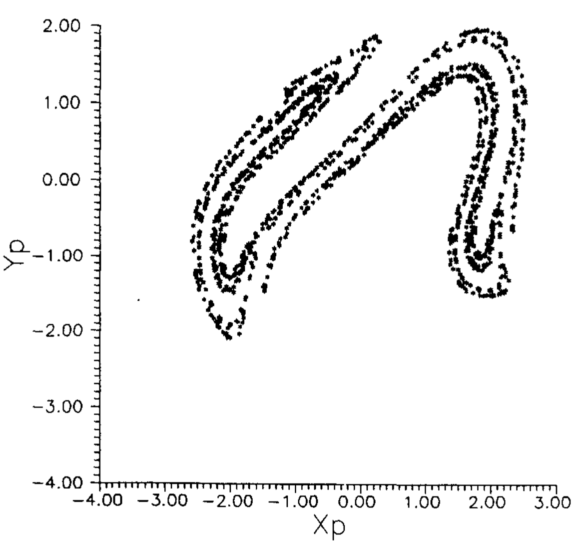


a) Nondimensional phase plane and Poincaré map

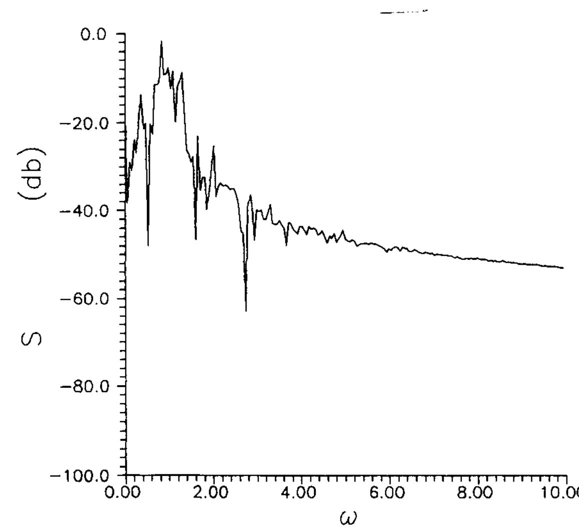


b) Power spectra

Fig. 6 Subharmonic ($m = 8$) solution [$\omega = 1.68$].

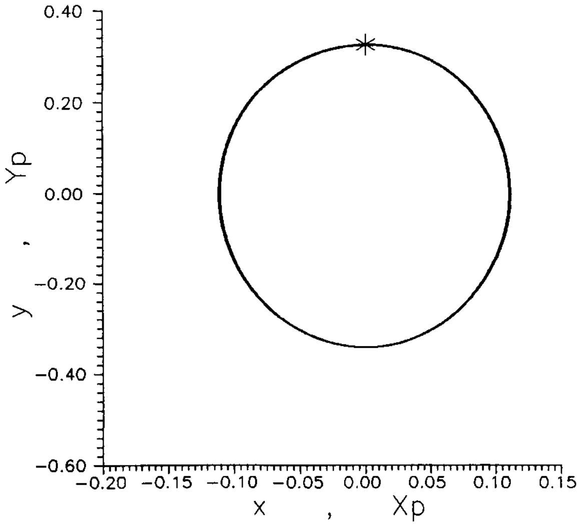


a) Nondimensional Poincaré map

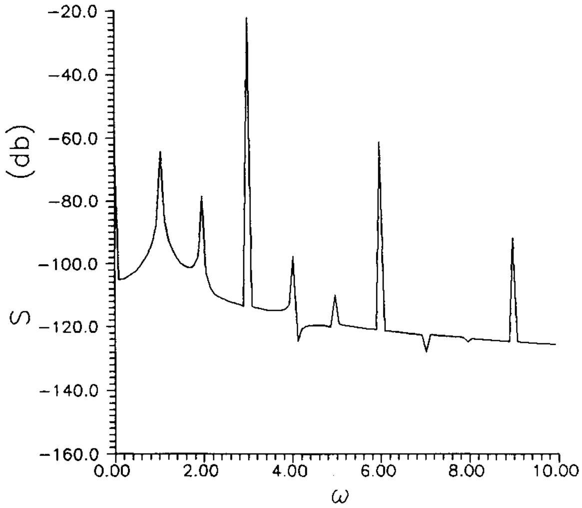


b) Power spectra

Fig. 7 Chaotic attractor [$\omega = 1.65$].

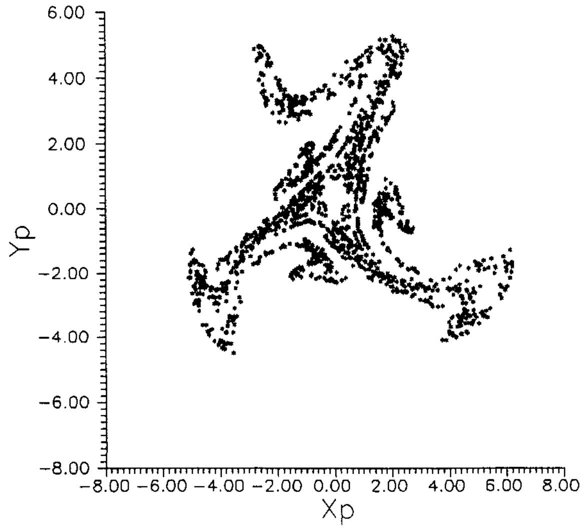


a) Nondimensional phase plane and Poincaré map

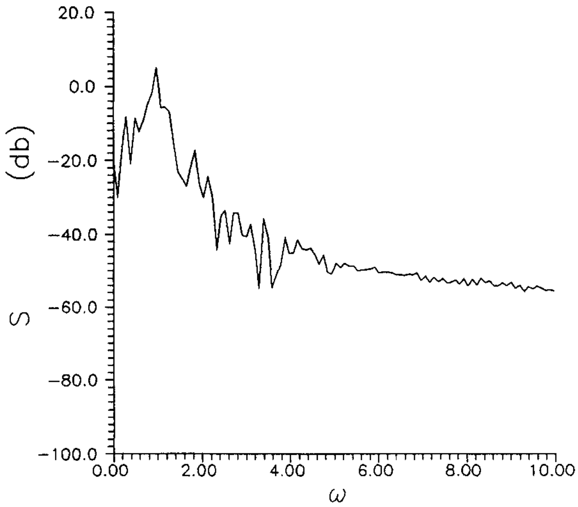


b) Power spectra

Fig. 8 Harmonic (m = 1) solution [ω = 3.05].



a) Nondimensional Poincaré map



b) Power spectra

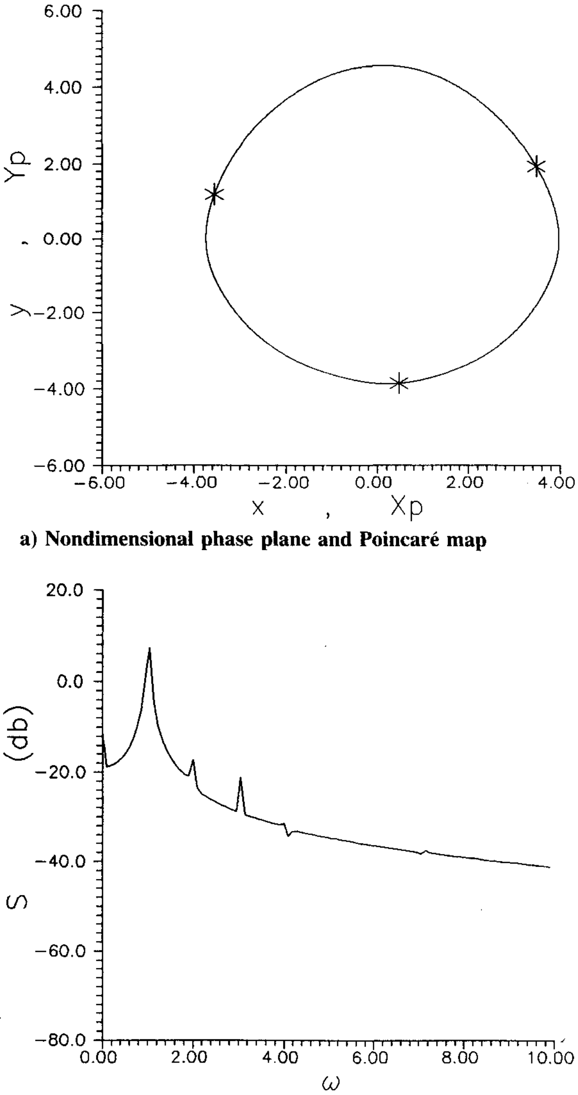
Fig. 9 Chaotic attractor [ω = 3.1].

regions defined by Eq. (12). Numerical integration of the system (μ = 1, δ = 0.1, f_1 = 0.1, γ = 0.01) with a linearized restoring force (α = 1, $\alpha_{n\geq3}$ = 0) and a negligible bias (f_0 = 0) reveals the existence of mT subharmonics (see Fig. 4, m = 2 and Fig. 10, m = 3) at the approximate stability limits (see Fig. 2). The results are portrayed with phase plane diagrams (x, y), Poincaré maps [Xp, Yp] where the mT subharmonic is depicted by m points and the power spectra [$S_x(\omega)$], where the order of the subharmonic is that of the peak with the largest energy content. Note that the magnitude (x, y) of both subharmonic responses is greater than that of the primary resonance.

Analysis of the $2T$ subharmonic variational equation [Eqs. (10a–10c), m = 2, j = 1,3] reveals further period doubling to $4T$. The approximate stability limit of the $4T$ solution can then be found by solving Eq. (10a) with the period doubled solution [Eq. (11), m = 2, j = 1,3]. Thus, the general Hill's equation suggests a possible cascade of period doubling bifurcations. If the period doubling sequence is infinite, the resulting motion is chaotic.[23] While the mT subharmonic repeats after m intervals, the chaotic (strange) attractor does not, consequently generating a fractal map (see Figs. 7a and 9a). The chaotic attractor is also characterized by a continuous spectra showing its "random like" behavior (see Figs. 7b and 9b).

The period doubling route to chaotic motion is continuous and is observed with the appearance of even harmonics (see Figs. 3–6). Another route to chaotic motion is observed in the abrupt change to and from neighboring periodic motions. This occurs near the local tangent bifurcation value and is characterized by contraction of the mT subharmonic (see Figs. 8a and 10a, m = 3). These routes can be described with evolution of unsymmetric (via a period doubling bifurcation) and symmetric (via a tangent bifurcation) solutions as is evident by the spectral content of the prechaotic and postchaotic motions (see Figs. 5b–7b and Figs. 8b–10b).

The symmetric and unsymmetric solutions are usually associated with symmetric and unsymmetric system nonlinearities[7] which are found in both drag and inertial components of the exciting force. However, combined parametric and external excitation exhibit chaotic subharmonic dynamics due to tangent bifurcations in a system with a quadratic nonlinearity[8] and generate period doubling bifurcations in a system with a symmetric nonlinearity.[18] Investigation of parameters influencing the instabilities of the system show that the period doubling route to chaos is sensitive to the magnitude of the



a) **Nondimensional phase plane and Poincaré map**

b) **Power spectra**

Fig. 10 Subharmonic ($m = 3$) solution [$\omega = 3.15$].

inertial force whereas the tangent route is controlled by the relationship between the restoring force and both drag and inertial components of the exciting force. The system with a linear restoring force is dominated by the inertial force ($\omega^2 f_1 > \delta$) which controls both tangent and period doubling routes to chaotic motion.

Summary and Conclusions

Stability analysis of a wave-structure interaction system has revealed the existence of local and global bifurcations identifying routes to chaotic motions. The system analyzed is characterized by a nonlinear restoring force and a coupled wave-structure exciting force consisting of quadratic drag and convective inertial components. The system which incorporates the exact form of the wave-structure nonlinearity is shown to exhibit properties of a complex nonlinear system subject to combined parametric and external forcing.

A semianalytic method describing local and global bifurcations has been derived and verified numerically. The method incorporates stability analysis of system response leading to stability curves of primary and secondary resonances in parameter space. Stability of approximate solutions is governed by a general Hill's equation which reveals existence of period doubling and tangent instabilities.

Local and global bifurcations were found by further analysis of the stability equations and reveal the possible existence of complex subharmonic and chaotic motions. Two routes to chaotic motion were identified evolving from the subharmonic period doubling and tangent bifurcations. The first is a continuous route via a period doubling cascade whereas the second is abrupt. Although these routes are typically associated with unsymmetric and symmetric system characteristics, the controlling mechanism found for a system with a linearized restoring force is the parametrically excited inertial force.

Large mooring system response with a linearized restoring force is characteristic of secondary resonant motions of single-point and weakly nonlinear multipoint (small angle) mooring ocean systems. The analysis reveals that although solutions remain bounded, enhanced large amplitude, subharmonic and unpredictable chaotic motions dominate system response even for moderate sea states. Consequently, control of transitions between coexisting states and sensitivity to initial conditions require exact identification of mechanisms governing system instabilities and routes to chaotic motion. The significance of this result is that nonlinear analysis of ocean mooring or similar mechanical systems neglecting the convective inertial force will not predict true system behavior and careful consideration should be given to model formulation prior to analysis and design stages.

Thus, analysis of a hydrodynamically excited nonlinear ocean system by a semianalytical method uncovers complex nonlinear system dynamics and the onset of chaotic motion allowing the identification of system instabilities and their generating mechanisms. Existence of chaotic dynamics, or sensitivity to initial conditions, reveals a loss of predictability in system behavior which cannot be obtained in systems described by an equivalently linearized exciting force. Therefore, in order to predict and control complex nonlinear and chaotic dynamics, an exact form of system nonlinearities is required for a comprehensive stability analysis of wave-structure interaction systems.

Appendix

Coefficients of the nonlinear function $F_1(x, \theta)$ and $F_2(x, y, \theta)$ of Eqs. (4b) and (4c):

$$F_1(x, \theta) = \sum_l [B_l + K_l(\theta)]x^l \tag{A1a}$$

where

$$B_0 = \mu\delta \operatorname{sgn}(u - y)\left[\left(\frac{f_0}{\omega}\right)^2 + \frac{1}{2}f_1^2\right]$$

$$B_{2l+1} = -\alpha_{2l+1}, \qquad B_{2l} = 0 \tag{A1b}$$

$$K_0 = \mu f_1\left\{\omega^2\left[-\left(1 - \frac{f_0}{\omega}\right)\sin\theta + \frac{1}{2}f_1 \sin 2\theta\right] + \operatorname{sgn}(u - y)\delta\left[2\frac{f_0}{\omega}\cos\theta - \frac{1}{2}f_1\cos 2\theta\right]\right\}$$

$$K_1 = \mu f_1\left\{\omega^2\left[\left(1 - \frac{f_0}{\omega}\right)\cos\theta + f_1\cos 2\theta\right] + \operatorname{sgn}(u - y)\delta\left[2\frac{f_0}{\omega}\sin\theta - f_1\cos 2\theta\right]\right\}$$

$$K_2 = \mu f_1\left\{\omega^2\left[\frac{1}{2}\left(1 - \frac{f_0}{\omega}\right)\sin\theta - f_1\sin 2\theta\right] + \operatorname{sgn}(u - y)\delta\left[-\frac{f_0}{\omega}\cos\theta + f_1\cos 2\theta\right]\right\}$$

$$K_3 = \mu f_1\left\{\omega^2\left[-\frac{1}{6}\left(1 - \frac{f_0}{\omega}\right)\cos\theta - \frac{2}{3}f_1\cos 2\theta\right] + \operatorname{sgn}(u - y)\delta\left[-\frac{1}{3}\frac{f_0}{\omega}\sin\theta - \frac{2}{3}f_1\sin 2\theta\right]\right\} \tag{A1c}$$

and

$$F_2(y, xy, \theta) = \sum_l \Gamma_l y^l + [\Lambda_l(\theta)x^l]y \qquad \text{(A2a)}$$

where

$$\Gamma_0 = 0, \qquad \Gamma_1 = -[\gamma + 2(f_0/\omega_2)]$$

$$\Gamma_2 = (1/\omega^2), \qquad \Gamma_{l\geq 3} = 0 \qquad \text{(A2b)}$$

$$\Lambda_0 = \mu f_1[-\omega \sin\theta - 2\,\mathrm{sgn}(u-y)(\delta/\omega)\cos\theta]$$

$$\Lambda_1 = \mu f_1[\omega\cos\theta - 2\,\mathrm{sgn}(u-y)(\delta/\omega)\sin\theta]$$

$$\Lambda_2 = (1/2)\mu f_1[\omega\sin\theta + 2\,\mathrm{sgn}(u-y)(\delta/\omega)\cos\theta]$$

$$\Lambda_3 = (1/6)\mu f_1[-\omega\cos\theta + 2\,\mathrm{sgn}(u-y)(\delta/\omega)\sin\theta] \qquad \text{(A2c)}$$

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